

# Composition of Roofs in Derived Category

SELCAN AKSOY  
 Florida State University  
 Faculty of Mathematics  
 E-mail: saksoy@math.fsu.edu

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## Abstract

In that paper, we prove that the composition of two roofs is another roof by using mapping cone of a morphism of cochain complexes.

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## 1 Introduction

Assume that  $\mathcal{A}$  is an abelian category. P. Aluffi defines a mapping cone  $MC(f)$  of a morphism  $f : A \rightarrow B$  in  $C(\mathcal{A})$  and homotopy between two morphisms in that category in [Al]. Also, in [Kr], H. Krause defines triangulated category and the localizing class. After that, he proves that the homotopic category  $K(\mathcal{A})$  is triangulated in Section 2.5.

Using this information, we prove that for a given upside down roof in the localization of  $K(\mathcal{A})$ , we can obtain a regular roof in that category. This allows us to compute the composition of two regular roofs in the localization of homotopic category.

## 2 Mapping Cone and Homotopy

The collection  $C(\mathcal{A})$  consisting of all cochain complexes in an abelian category  $\mathcal{A}$  forms an abelian category. It is easy to show that the set of morphisms of that category is an abelian group, finite products and coproducts exist since they exist in  $\mathcal{A}$ .

**Definition 2.0.1.** A morphism  $f$  between cochain complexes is quasi isomorphism if it induces an isomorphism in cohomology.

For a given morphism  $f : A \rightarrow B$  between cochain complexes  $A$  and  $B$ , we define a mapping cone  $MC(f)$  as  $MC(f)^i = A[1]^i \oplus B^i = A^{i+1} \oplus B^i$  for all  $i$ . Here, we get the morphisms

$$\begin{aligned} d_{MC(f)}^i : MC(f)^i &\rightarrow MC(f)^{i+1}, \\ d_{MC(f)}^i(a, b) &= (-d_A^{i+1}(a), f^{i+1}(a) + d_B^i(b)) \end{aligned}$$

between those objects.

$MC(f)$  is a cochain complex since  $d_{MC(f)}^{i+1} \circ d_{MC(f)}^i = 0$ .

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{i+1} & \xrightarrow{-d_A^{i+1}} & A^{i+2} & \xrightarrow{-d_A^{i+2}} & A^{i+3} \longrightarrow \cdots \\ & & \oplus & \searrow f^{i+1} & \oplus & \searrow f^{i+2} & \oplus \\ \cdots & \longrightarrow & B^i & \xrightarrow{d_B^i} & B^{i+1} & \xrightarrow{d_B^{i+1}} & B^{i+2} \longrightarrow \cdots \end{array}$$

**Definition 2.0.2.** A homotopy  $k$  between two morphisms of cochain complexes  $f, g : A \rightarrow B$  is a collection of morphisms  $k^i : A^i \rightarrow B^{i-1}$  such that for all  $i$ ,

$$g^i - f^i = d_B^{i-1} \circ k^i + k^{i+1} \circ d_A^i.$$

The above morphisms  $f$  and  $g$  are homotopic if there is a homotopy between them. We use the following diagram to show that homotopy and use the symbol  $f \sim g$  to mean there exists a homotopy between the morphisms  $f$  and  $g$ .

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} \longrightarrow \cdots \\ & & \swarrow k^{i-1} & & \swarrow k^i & & \swarrow k^{i+1} \\ \cdots & \longrightarrow & B^{i-2} & \xrightarrow{d_B^{i-2}} & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i \longrightarrow \cdots \end{array}$$

**Definition 2.0.3.** A morphism  $f : A \rightarrow B$  is a homotopy equivalence if there is a morphism  $g : B \rightarrow A$  such that  $f \circ g \sim id_B$  and  $g \circ f \sim id_A$ .

$A$  and  $B$  are homotopy equivalent if there is a homotopy equivalence  $A \rightarrow B$ .

**Proposition 2.0.1.** [\[Al\]](#) If  $f, g : A \rightarrow B$  are homotopic, then  $H^\bullet(f) = H^\bullet(g)$ .

**Corollary 2.0.1.** If  $f : A \rightarrow B$  is homotopy equivalence, then  $H^\bullet(A) \cong H^\bullet(B)$ .

Every homotopy equivalence is a quasi isomorphism, but every quasi isomorphism may not be a homotopy equivalence.

### 3 Triangulated Categories

**Definition 3.0.4.** [\[Kr\]](#) Assume that  $\mathcal{A}$  is an additive category with an equivalence  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$ . A triangle  $(f, g, h)$  in  $\mathcal{A}$  is a sequence of morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \mathcal{F}(X)$  for all objects  $X, Y$  and  $Z$  in  $\mathcal{A}$ .

A morphism between two triangles  $(f_1, g_1, h_1)$  and  $(f_2, g_2, h_2)$  is a triple  $(k_1, k_2, k_3)$  of morphisms in  $\mathcal{A}$  making the following diagram commute.

$$\begin{array}{ccccccc} X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & Z & \xrightarrow{h_1} & \mathcal{F}(X) \\ k_1 \downarrow & & k_2 \downarrow & & k_3 \downarrow & & \mathcal{F}(k_1) \downarrow \\ X' & \xrightarrow{f_2} & Y' & \xrightarrow{g_2} & Z' & \xrightarrow{h_2} & \mathcal{F}(X') \end{array}$$

The category  $\mathcal{A}$  is called pre-triangulated if it has a class of exact triangles satisfying the following conditions.

1. A triangle is exact if it is isomorphic to an exact triangle.
2. For all objects  $X$  in  $\mathcal{A}$ , the triangle  $0 \longrightarrow X \xrightarrow{id} X \longrightarrow 0$  is exact.
3. Each morphism  $f : X \rightarrow Y$  can be completed to an exact triangle  $(f, g, h)$ .
4. A triangle  $(f, g, h)$  is exact if and only if the triangle  $(g, h, -\mathcal{F}(f))$  is exact.
5. Given two exact triangles  $(f_1, g_1, h_1)$  and  $(f_2, g_2, h_2)$ , each pair of maps  $k_1$  and  $k_2$  satisfying  $k_2 \circ f_1 = f_2 \circ k_1$  can be completed to a morphism;

$$\begin{array}{ccccccc} X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & Z & \xrightarrow{h_1} & \mathcal{F}(X) \\ k_1 \downarrow & & k_2 \downarrow & & k_3 \downarrow & & \mathcal{F}(k_1) \downarrow \\ X' & \xrightarrow{f_2} & Y' & \xrightarrow{g_2} & Z' & \xrightarrow{h_2} & \mathcal{F}(X') \end{array}$$

$\mathcal{A}$  is a triangulated category if in addition it satisfies the following axiom.

6. **The Octahedral Axiom:** Given exact triangles  $(f_1, f_2, f_3)$ ,  $(g_1, g_2, g_3)$  and  $(h_1, h_2, h_3)$  with  $h_1 = g_1 \circ f_1$ , there exists an exact triangle  $(k_1, k_2, k_3)$  making the following diagram commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & U & \xrightarrow{f_3} & \mathcal{F}(X) \\
\downarrow = & & \downarrow g_1 & & \downarrow k_1 & & \downarrow = \\
X & \xrightarrow{h_1} & Z & \xrightarrow{h_2} & V & \xrightarrow{h_3} & \mathcal{F}(X) \\
& & \downarrow g_2 & & \downarrow k_2 & & \downarrow \mathcal{F}(f_1) \\
& & W & \xrightarrow{=} & W & \xrightarrow{g_3} & \mathcal{F}(Y) \\
& & \downarrow g_3 & & \downarrow k_3 & & \\
& & \mathcal{F}(Y) & \xrightarrow{\mathcal{F}(f_2)} & \mathcal{F}(U) & & 
\end{array}$$

*Remark 3.0.1.* If  $\mathcal{A}$  is a pretriangulated category, then  $A^{op}$  is a pretriangulated category, too.

## 4 The Localization of A Category

**Definition 4.0.5.** [Kr] Assume that  $\mathcal{A}$  is a category and  $F$  is a class of maps in  $\mathcal{A}$ .  $F$  is a localizing class if the following conditions are satisfied.

1. If  $f, g$  are composable maps in  $F$ , then  $g \circ f$  is in  $F$ .
2. The identity map  $id_A$  is in  $F$  for all  $A \in \mathcal{A}$ .
3. If  $f : A \rightarrow B$  is in  $F$ , then every pair of maps  $B' \rightarrow B$  and  $A \rightarrow A''$  in  $\mathcal{A}$  can be completed to a pair of commutative diagrams;

$$\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow f' & & \downarrow f \\
B' & \longrightarrow & B
\end{array}
\quad
\begin{array}{ccc}
A & \longrightarrow & A'' \\
\downarrow f & & \downarrow f'' \\
B & \longrightarrow & B''
\end{array}$$

such that  $f'$  and  $f''$  are in  $F$ .

4. If  $f, g : A \rightarrow B$  are maps in  $\mathcal{A}$ , then there is some  $h : A' \rightarrow A$  in  $F$  with  $f \circ h = g \circ h$  if and only if there is some  $k : B \rightarrow B''$  in  $F$  with  $k \circ f = k \circ g$ .

**Definition 4.0.6.** [Kr] Assume that  $\mathcal{A}$  is a category and  $F$  is a class of maps in  $\mathcal{A}$ . The localization of  $\mathcal{A}$  with respect to  $F$  is a category  $\mathcal{A}[F^{-1}]$  together with a functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}[F^{-1}]$  such that  $\mathcal{F}(f)$  is an isomorphism for all  $f$  in  $F$  and any functor  $\mathcal{G} : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\mathcal{G}(f)$  is an isomorphism for all  $f$  in  $F$  factors uniquely through  $\mathcal{F}$ .

We can always find a localization like that.

**Definition 4.0.7.** Assume that  $\mathcal{A}$  is a category and  $F$  is a localizing class. The objects of  $\mathcal{A}[F^{-1}]$  are the objects of  $\mathcal{A}$ . The morphisms  $A \rightarrow B$  in  $\mathcal{A}[F^{-1}]$  are equivalence classes of diagrams  $A \xleftarrow{f} B' \xrightarrow{g} B$  with the morphism  $f$  in  $F$  for all objects  $A$  and  $B$  in the category  $\mathcal{A}[F^{-1}]$ . We will call those morphisms as regular roofs.

A pair  $(f, g)$  is also called a fraction because it is written as  $g \circ f^{-1}$  in  $\mathcal{A}[F^{-1}]$ .

*Remark 4.0.2.* The functor  $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}[F^{-1}]$  sends a map  $f : A \rightarrow B$  to the pair  $(id_A, f)$ .

**Definition 4.0.8.**  $(f, g)$  and  $(f', g')$  are equivalent if there exists a commutative diagram with  $f''$  in  $F$ ;

$$\begin{array}{ccccc}
 & & B' & & \\
 & f \nearrow & \uparrow & \nwarrow g & \\
 A & \xleftarrow{f''} & B''' & \xrightarrow{g''} & B \\
 & f' \nwarrow & \downarrow & \nearrow g' & \\
 & & B'' & & 
 \end{array}$$

## 5 Composition of Two Roofs

**Definition 5.0.9.** [Al] Assume that  $\mathcal{A}$  is an abelian category.  $K(\mathcal{A})$  is a category whose objects are the objects in  $C(\mathcal{A})$  and the set of morphisms is

$$Hom_{K(\mathcal{A})}(A, B) = Hom_{C(\mathcal{A})}(A, B) / \sim$$

where  $\sim$  is homotopy relation.

If  $f \circ g \sim id$  in  $C(\mathcal{A})$ , then  $f \circ g = id$  in  $K(\mathcal{A})$ . As a result, homotopy equivalences in  $C(\mathcal{A})$  become isomorphisms in  $K(\mathcal{A})$  and we say that  $K(\mathcal{A})$  is obtained by inverting all homotopy equivalences in  $C(\mathcal{A})$ . It is an additive category, but not abelian in general since homotopic maps don't have same kernels and cokernels.

In [Kr], H. Krause proves that  $K(\mathcal{A})$  is a triangulated category.

*Remark 5.0.3.* The set of quasi isomorphisms in  $K(\mathcal{A})$  for a given abelian category  $\mathcal{A}$  forms a localizing class.

**Theorem 5.0.2.** [Al] Assume that  $\mathcal{A}$  is an abelian category and we have two morphisms  $L \xrightarrow{\alpha} \underline{K}$  and  $M \xrightarrow{\beta} \underline{K}$  with  $\beta$  is a quasi isomorphism for objects  $L$  and  $M$  in  $K(\mathcal{A})$ . Then, there exists a cochain complex  $K$ , morphisms  $K \rightarrow L$  which is quasi isomorphism

and  $K \rightarrow M$  in  $K(\mathcal{A})$  such that the following diagram commutes.

$$\begin{array}{ccc}
 & K & \\
 \gamma_2 \swarrow & & \searrow \gamma_1 \\
 L & & M \\
 \alpha \searrow & & \swarrow \beta \\
 & \underline{K} &
 \end{array} \tag{1}$$

*Proof.* Assume that  $\gamma$  is the composition  $L \longrightarrow \underline{K} \longrightarrow MC(\beta)$ ,  $K = MC(\gamma)[-1]$  and  $K^i = L^i \oplus M^i \oplus \underline{K}^{i-1}$ . We define morphisms  $K^i \rightarrow L^i$ ,  $(l, m, \underline{k}) \rightarrow l$  and  $K^i \rightarrow M^i$ ,  $(l, m, \underline{k}) \rightarrow -m$  as in [AI].

We want to prove that  $L$  and  $M$  are connected by a regular roof as well.

For the rest of the proof, we need to show that the **Diagram 1** commutes.

$H^\bullet(M) \cong H^\bullet(\underline{K})$  since  $\beta$  is a quasi isomorphism. This implies that  $MC(\beta)$  is exact, so  $H^\bullet(MC(\beta)) = 0$ .

$$MC(\beta)^i = M[1]^i \oplus \underline{K}^i = M^{i+1} \oplus \underline{K}^i,$$

$$d_{MC(\beta)}^i : MC(\beta)^i \rightarrow MC(\beta)^{i+1}, \quad d_{MC(\beta)}^i(m, \underline{k}) = (-d_M^{i+1}, \beta^{i+1}(m) + d_{\underline{K}}^i(\underline{k})).$$

We define  $\gamma^i(l) = (0, \alpha^i(l))$  and

$$MC(\gamma)^i = L[1]^i \oplus M^{i+1} \oplus \underline{K}^i = L^{i+1} \oplus M^{i+1} \oplus \underline{K}^i$$

where  $d_{MC(\gamma)}^i : MC(\gamma)^i \rightarrow MC(\gamma)^{i+1}$  with

$$\begin{aligned}
 d_{MC(\gamma)}^i(l, m, \underline{k}) &= (-d_L^{i+1}(l), \gamma^{i+1}(l) + d_{MC(\beta)}^i(m, \underline{k})) = \\
 &= (-d_L^{i+1}(l), -d_M^{i+1}(m), \alpha^{i+1}(l) + \beta^{i+1}(m) + d_{\underline{K}}^i(\underline{k})).
 \end{aligned}$$

$$K = MC(\gamma)[-1],$$

$$MC(\gamma)[-1]^i = MC(\gamma)^{i-1} = L^i \oplus M^i \oplus \underline{K}^{i-1} = K^i$$

and  $d_K^i = -d_{MC(\gamma)}^{i-1}$  with

$$d_K^i = (d_L^i(l), d_M^i(m), -\alpha^i(l) - \beta^i(m) - d_{\underline{K}}^{i-1}(\underline{k})).$$

Assume that  $h^i : K^i \rightarrow \underline{K}^{i-1}$  takes  $(l, m, \underline{k})$  to  $-\underline{k}$ . We need to show that

$$\alpha^i \circ \gamma_2^i - \beta^i \circ \gamma_1^i = d_{\underline{K}}^{i-1} \circ h^i + h^{i+1} \circ d_K^i$$

for all  $i \in \mathbb{Z}$  which shows that  $\alpha \circ \gamma_2$  and  $\beta \circ \gamma_1$  are homotopic maps in  $K(\mathcal{A})$ . This will show that they are same maps.

For all  $(l, m, \underline{k}) \in K^i$ ,

$$\begin{aligned} (\alpha^i \circ \gamma_2^i - \beta^i \circ \gamma_1^i)(l, m, \underline{k}) &= \alpha^i(\gamma_2^i(l, m, \underline{k})) - \beta^i(\gamma_1^i(l, m, \underline{k})) \\ &= \alpha^i(l) - \beta^i(-m) = \alpha^i(l) + \beta^i(m) \end{aligned}$$

since  $\mathcal{A}$  is additive. On the other hand,

$$\begin{aligned} (d_{\underline{K}}^{i-1} \circ h^i + h^{i+1} \circ d_K^i)(l, m, \underline{k}) &= d_{\underline{K}}^{i-1}(h^i(l, m, \underline{k})) + h^{i+1}(d_N^i(l, m, \underline{k})) \\ &= d_{\underline{K}}^{i-1}(-k) + \alpha^i(l) + \beta^i(m) + d_{\underline{K}}^{i-1}(\underline{k}) = \alpha^i(l) + \beta^i(m). \end{aligned}$$

This shows the maps are homotopic and the diagram is commutative.

We need to show that  $\gamma_2$  is a quasi isomorphism. We have an exact triangle;

$$\begin{array}{ccc} & L & \\ \nearrow & & \searrow \\ L[1] + MC(\beta) & \xleftarrow{\quad} & MC(\beta) \end{array}$$

This triangle is isomorphic to an exact triangle;

$$\begin{array}{ccc} & MC(\beta) & \\ \nearrow & & \searrow \\ L[1] & \xleftarrow{\quad} & L[1] + MC(\beta) \end{array}$$

Then, we take its cohomology and the triangle still will be exact.

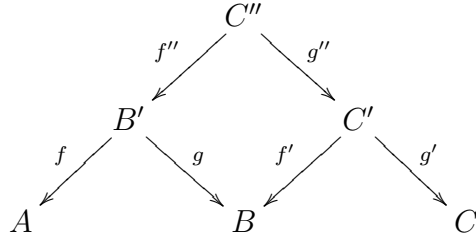
$$\begin{array}{ccc} & H^\bullet(MC(\beta)) & \\ \nearrow & & \searrow \\ H^\bullet(L[1]) & \xleftarrow{\quad} & H^\bullet(L[1] + MC(\beta)) \end{array}$$

$H^\bullet(MC(\beta)) = 0$ , so  $H^\bullet(L[1]) \cong H^\bullet(L[1] + MC(\beta))$  and

$$L[1] + MC(\beta) = L[1] + M[1] + \underline{K} = K[1].$$

Consequently,  $H^\bullet(L[1]) \cong H^\bullet(K[1])$ . This means  $H^\bullet(L) \cong H^\bullet(K)$ , hence  $\gamma_2$  is a quasi isomorphism.  $\square$

The pair  $(f \circ f'', g' \circ g'')$  is the composition of two pairs  $(f, g)$  and  $(f', g')$  as in the following commutative diagram;



## References

- [Al] P. Aluffi, **Algebra: Chapter 0**, Graduate Studies in Mathematics, AMS, 104 (2009), 512-dc22.
- [Kr] H. Krause, **Derived Categories, Resolutions, and Brown Representability**, arxiv: math/ 0511047v3, 2006.